Topics in PDE - List 6

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The reference for this list is [1] and the lecture notes.
Let ℓ^2 be the set of all sequences $z = \{z_n\}_{n\in\mathbb{N}}$ in $\mathbb{C}$ such that $\sum_{n=1}^{\infty} \|z_n\|^2 < \infty$.

Exercise 1. *For a fixed $j \in \mathbb{N}$, consider the operator $K_j : \ell \to \ell$ given by*

$$(Kz)_n \doteq x_{n+j}, \qquad n \in \mathbb{N}.$$

1. *Is K a compact operator? Justify your answer.*
2. *Is K of finite index? Justify your answer.*

Exercise 2. *Let K be a compact operator on a Hilbert space H.*

1. *Show that the spectrum $\sigma(K)$ is at most countable.*
2. *Show that $\sigma(K)\backslash\{0\}$ consists of (generilized) eigenvalues with finite multiplicity.*
3. *Find an example where 0 is in $\sigma(K)$, but it is not an eigenvalue of K.*

Exercise 3. *Let K be a compact self-adjoint operator on a Hilbert space H.*

1. *Show that the multiplicity of $\lambda \in \sigma(K)\backslash\{0\}$ is 1.*
2. *Show that $\sigma(K) \subset \mathbb{R}$.*

Exercise 4. *Let $K : E \to E$ a compact operator and E a Banach space. Define $A = I - \lambda^{-1}K$ for $\lambda \in \sigma(K)$. Show that there exists an integer r such that $\ker A^n = \ker A^r$ if $n \geq r$.*

Exercise 5. *Let $I = [0, 1]$, and let $K \in L^2(I \times I)$. Define the operator $T : L^2(I) \to L^2(I)$ by*

$$(Tx)(s) = \int_0^1 K(s,t)x(t)\,\mathrm{d}t, \qquad s \in I.$$

1. *Show that T is compact;*
2. *Show that if $K(s,t) = \overline{K(s,t)}$ for almost every $s, t \in I$, then T is self-adjoint;*

References

[1] Gerald B. Folland. *Introduction to partial differential equations*. Princeton University Press, Princeton, NJ, second edition, 1995.