

Topics in PDE - List 4

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The reference for this list is [1] and the lecture notes.

Exercise 1. *Let $a < b$ be two real numbers, and consider H to be the set of all functions defined on $[a, b]$ with real (or complex) values. Assume that all functions are equibounded at $x_0 \in [a, b]$ and differentiable with equibounded derivatives at x_0 . Show that H is equicontinuous at x_0 .*

Exercise 2. *Let E be a compact space and F a metric space, and consider H to be a subset of $C(E, F)$. Show that if H is compact, then it is equicontinuous.*

Exercise 3. *Show, using an example in low dimensions, that the heat operator is not analytic-hypoelliptic.*

Exercise 4. *Show, using an example in low dimensions, that the wave operator is not hypoelliptic.*

For the following exercises, use Folland as a reference, and assume that Ω is a bounded domain in \mathbb{R}^n .

Exercise 5. 1. *Write the definition of the Sobolev spaces $H^k(\Omega)$ for integer k , and define the norms $\|\cdot\|_{j,\Omega}$.*

2. *Let $j \leq k$ be integers. Show that $\|\cdot\|_{j,\Omega} \leq \|\cdot\|_{k,\Omega}$ and that $H^k(\Omega)$ is contained in $H^j(\Omega)$ as a dense subspace.*

Exercise 6. 1. *Show that if $\phi \in C^\infty(\overline{\Omega})$, the map $f \mapsto \phi f$ is bounded on $H^k(\Omega)$ for all k .*

2. *Show that the restriction map $f \mapsto f|_\Omega$ is bounded from $H^k(\mathbb{R}^n)$ to $H^k(\Omega)$.*

References

- [1] Gerald B. Folland. *Introduction to partial differential equations*. Princeton University Press, Princeton, NJ, second edition, 1995.