Topics in PDE - List 5

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The reference for this list is [1] and the lecture notes.

Let L be a strongly elliptic second-oder partial differential operator with $C^{l+\alpha}$ coefficients (see the precise form in the statement of the Hopf Principle in the lecture notes), and let $f \in C^{l+2+\alpha}(\overline{\Omega})$.

Exercise 1. *Assume that for $h \in C^{l+2+\alpha}(\overline{\Omega})$, there exists a constant $C > 0$ such that*

$$\|h\|_{l+2+\alpha} \leq C(\|Lh\|_{l+\alpha} + \|h\|_{(0)} + \|h|_{\delta\Omega}\|)_{l+2+d}$$

(these are the Schauder estimates). Show that if $h = 0$ in $\delta\Omega$, then

$$\|h\|_{l+2+\alpha} \leq C\|Lh\|_{l+\alpha}.$$

We denote by $(P)(L)$ the Dirichlet problem $Lu = f$ with $u = \phi$ in $\partial\Omega$, where $\Omega \subset \mathbb{R}^n$ is an open, bounded set.

Exercise 2. *Show that, to solve the Dirichlet $Lu = f$ with $u = \phi$ in $\delta\Omega$, it is enough to solve the problem $Lv = f - Lg$ with $v = 0$ in $\delta\Omega$ where $g \in C^{2+\alpha}(\overline{\Omega})$ in which $g = \phi$ in $\delta\Omega$.*

Exercise 3. *Consider the homotopy $L_t = tL + (1-t)\Delta$ for $t \in [0, 1]$. Let $J \subset [0, 1]$ be the set of all t for which $(P)(L_t)$ has a solution. For $t_0 \in J$ and $t \in [0, 1]$, define*

$$v = A_t u \in C^{2+\alpha}(\overline{\Omega})$$

the unique solution to the problem

$$L_{t_0} = (L_{t_0} - L_t)u + f$$

with $u = 0$ in $\delta\Omega$. Assume that there exists $\rho > 0$ such that $|t - t_0| < \rho$, then $A_t : C^{2+\alpha}(\overline{\Omega}) \to C^{2+\alpha}(\overline{\Omega})$ is a contraction and conclude that $t \in J$.

Exercise 4. *Let E be a Banach space and $K : E \to E$ a compact operator. Show that the (Fredholm) index of $I - K$ is zero.*

References

[1] Gerald B. Folland. *Introduction to partial differential equations.* Princeton University Press, Princeton, NJ, second edition, 1995.