Topics in PDE - List 3

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The reference for this list is [1] and the lecture notes.

Exercise 1. *Let $\Omega \subset \mathbb{R}^n$ be an open set, and denote by $H(\Omega)$ the set of all harmonic functions defined on Ω.*

1. *Show that $H(\Omega) \subset C^\infty(\Omega)$.*

   Hint*: Use an integral representation of harmonic functions and differentiation under the integral sign.*

2. *Show that $H(\Omega) \subset C^\omega(\Omega)$.*

   Hint*: use the item above to prove that, if $f \in H(\Omega)$, and $K \subset\subset \Omega$, then there are constants $M, C > 0$ such that $\sup_{x \in K} |\partial^\alpha f(x)| \leq MC^{|\alpha|}|\alpha|!$ for all $\alpha \in \mathbb{Z}^n_+$ and conclude that $f \in C^\omega(\Omega)$.*

The precise statements and proofs of the following results can be found in Folland's book.

Exercise 2. *Prove the Sobolev Lemma.*

Exercise 3.

1. *Write the precise definition of a compact operator.*
2. *Prove Rellich's Theorem.*

Exercise 4. *Let $P = P(x, D)$ be an elliptic operator of order $2m$. Show that*

$$\operatorname{Ker} P = \{u \in L^2(\Omega) : Pu = 0\}$$

is finite-dimensional.

Exercise 5. *Let $P = P(x, D)$ be an elliptic operator of order $2m$.*

1. *State the Spectral Theorem as it appears in* Functional Analysis*.*
2. *Assume that P is self-adjoint and compact. Use the Spectral Theorem to obtain an orthonormal basis of eigenfunctions of P in $L^2(\Omega)$.*

Exercise 6. *Let Δ be the Laplacian in the unitary disc $D(0, 1) = \{x \in R^2\}$*

References

[1] Gerald B. Folland. *Introduction to partial differential equations.* Princeton University Press, Princeton, NJ, second edition, 1995.