Topics in PDE - List 2

Max Reinhold Jahnke
max.jahnke@uni-koeln.de

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The reference for this list is [1] and the lecture notes.

Exercise 1. *In the following, assume the same hypothesis as in Harnack's inequality and take* $K = \overline{B(x_0, r)} \subset B$.

- *Show that*
$$u(x')\left(\frac{R-r}{R+r}\right) \leq u(x) \leq u(x')\left(\frac{R+r}{R-r}\right)$$
for all $x, x' \in K$*;*

- *Show that, if* $u = 0$ *at some point, then* $u \equiv 0$.

Exercise 2. *Let* u *be a harmonic function in* $\mathbb{R}^n$*. Show that if* $\sup u < \infty$ *or* $\inf u > -\infty$*, then* $u \equiv 0$.

Exercise 3. *Let* Ω *be an open bounded set and assume that* $x \in \partial\Omega$ *is an isolated point of* $\partial\Omega$*. Show that there are no barriers for* Ω *at* x.

Exercise 4. *Let* Ω *be an open bounded set and assume that* $\partial\Omega$ *is a* C^2 *manifold. Show that all points of* $\partial\Omega$ *admit a barrier.*

Exercise 5. *Consider the maps* H_- *and* H_+ *defined in the lecture on the Perron method. Prove the inequalities*

$$H_-(f) + H_+(f) \leq H_-(f+g) \leq H_+(f+g) \leq H_+(f) + H_+(g)$$

for all f, g *bounded functions defined on* $\partial\Omega$.

Exercise 6. *Prove directly that if* p *is a polynomial, then*

$$H_-(p) = H_+(p).$$

References

[1] Gerald B. Folland. *Introduction to partial differential equations.* Princeton University Press, Princeton, NJ, second edition, 1995.