Topics in PDE - List 1

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The reference for this list is [1].

Exercise 1. *Let $\Omega \subset \mathbb{R}^n$ be an open set.*

- *What is the definition of a distribution in Ω?*
- *Show that $\delta : C_c^\infty(\Omega) \to \mathbb{C}$, the functional defined by $\delta_x(\phi) = \phi(x)$ for $x \in \Omega$, is a distribution.*

Exercise 2. *Consider the function E defined in $\mathbb{R}^n \backslash \{0\}$.*

$$E(x) = \begin{cases} \dfrac{1}{(2-n)|S^{n-1}|} \dfrac{1}{|x|^{n-2}}, & n \geq 3, \\ \dfrac{1}{2\pi} log|x|, & n = 2. \end{cases} \tag{1}$$

Here $|S^{n-1}|$ denotes the "area" of the sphere $S^{n-1} = \{x \in \mathbb{R}^n; |x| = 1\}$.

- *Show that $\Delta E = 0$ in $\mathbb{R}^n \backslash \{0\}$;*
- *Show that $\Delta E = \delta_0$ in any open set $\Omega \subset \mathbb{R}^n$ such that $0 \in \Omega$.*

Exercise 3.

1. *State (without proof) the Divergence Theorem;*
2. *State and prove the Green's identity for $B(x, r)$.*

Exercise 4. *Let u be harmonic an open set Ω and let $B(x, r) \subset\subset \Omega$.*

- *Use the Green identities to show that*

$$u(x) = \frac{1}{r^{n-1}|S^{n-1}|} \int_{S(x,r)} u(y)\, \mathrm{d}\sigma(y)$$

Exercise 5. *Write the Poisson formula for the ball in $\mathbb{R}^n$ and for the disc in $\mathbb{R}^2$.*

Exercise 6.

- *State the maximum modulus principle for harmonic functions;*
- *Use the maximum modulus principle to prove uniqueness for the solution of the Dirichlet problem for the Laplacian, that is, show that you cannot have two distinct solutions $u \in C^2(\Omega) \cap C^0(\overline{\Omega})$ for the problem*

$$\begin{cases} \Delta u = 0, & in\ \Omega; \\ u = u_0, & in\ \partial\Omega. \end{cases}.$$

References

[1] Gerald B. Folland. *Introduction to partial differential equations.* Princeton University Press, Princeton, NJ, second edition, 1995.